Jan 1999

Day 2, Q2

Consider the expansion

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n$$

(a) Prove that the coefficients c_n satisfy $c_n = c_{n-1} + c_{n-2}$, $n \geq 2$.

Solution. Rewrite the equality above as

$$\begin{aligned} 1 &= (1-z-z^2)\sum_{n=0}^{\infty} c_n z^n \\ &= c_0 + (c_1 - c_0)z + \sum_{n=2}^{\infty}(c_n - c_{n-1} - c_{n-2})z^n \end{aligned}$$

This implies then, in order for the equality above to hold

$$\begin{aligned} c_0 &= 1 \\ c_1 - c_0 &= 0 \\ c_n - c_{n-1} - c_{n-2} &= 0 \ \forall n \geq 2 \end{aligned}$$

(b) Find the radius of convergence of the series.

Solution. We would like to view the function on the left as a function of a complex variable. In this case, we know that the function has a convergent power series expansion from the origin of the complex plane to the nearest pole.

The roots of the equation $1 - z - z^2 = 0$ are $z_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. The nearest pole lies along the circle of radius $R = \frac{\sqrt{5}-1}{2}$. This is the radius of convergence.

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Day 2, Q3

Derive a two-point Gaussian quadrature to approximate the integral

$$\int_0^{\infty} f(x)e^{-x}dx$$

for bounded, C^{∞} functions $f(x)$ on $[0, \infty)$.

Solution. We would like to derive a set of orthogonal polynomials using an L^2 norm weighted with the function e^{-x} and then evaluate the function $f(x)$ at the roots of the n^{th} polynomial.

For this problem, $n = 2$. Start with the standard definition, $P_0(x) = 1$ ($\int_0^\infty P_0^2(x)e^{-x}dx = 1$). Now, we want a $P_1(x) = ax + b$ s.t.

$$\begin{aligned} \int_0^\infty P_1^2(x)e^{-x}dx &= 1 \text{ and} \\ \int_0^\infty P_1(x)P_2(x)e^{-x}dx &= 0. \end{aligned}$$

Carrying out the integration

$$\begin{aligned} \int_0^\infty P_1^2(x)e^{-x}dx &= \int_0^\infty (a^2x^2 + 2abx + b^2)e^{-x}dx \\ &= 2a^2 + 2ab + b^2 = 1 \\ \int_0^\infty P_1(x)P_0(x)e^{-x}dx &= \int_0^\infty (ax + b)e^{-x}dx \\ &= a + b = 0 \end{aligned}$$

which leads to the system

$$\begin{aligned} a &= -b \\ \Longrightarrow 2b^2 - 2b^2 + b^2 &= 1 \\ \Longrightarrow b = -1, a &= 1 \end{aligned}$$

So, $P_1(x) = x - 1$. To find $P_2(x) = ax^2 + bx + c$, we evaluate the integrals

$$\begin{aligned} \int_0^\infty P_2^2(x)e^{-x}dx &= 12a^2 + 12ab + 4ac + 2b^2 + 2bc + c^2 = 1 \\ \int_0^\infty P_2(x)P_1(x)e^{-x}dx &= 6a + 2(b - a) + (c - b) - c = 0 \\ \int_0^\infty P_2(x)P_1(x)e^{-x}dx &= 2a + b + c = 0 \end{aligned}$$

The solution of which is $a = \frac{1}{2}, b = -2, c = 1$. So, $P_2(x) = \frac{1}{2}\left(x^2 - 4x + 2\right)$.

The roots of $P_2(x)$ are $x_{1,2} = 2 \pm \sqrt{2}$. These are the nodes for Gaussian quadrature. Now, we need to find the associated weights.

We need the Gaussian quadrature to be exact for $f(x) = 1$ and $f(x) = x$ (and higher order x^n if we desired more points to the interpolation, but we only need 2 to solve the linear system).

$$\begin{aligned} \int_0^\infty 1e^{-x}dx &= Af(2 + \sqrt{2}) + Bf(2 - \sqrt{2}) = A + B = 1 \\ \int_0^\infty xe^{-x}dx &= Af(2 + \sqrt{2}) + Bf(2 - \sqrt{2}) = A(2 + \sqrt{2}) + B(2 - \sqrt{2}) = 1 \end{aligned}$$

Solving for A, B yields

$$\begin{aligned} A &= \frac{2-\sqrt{2}}{4} \\ B &= \frac{2+\sqrt{2}}{4} \end{aligned}$$

So the two-point Gaussian quadrature is

$$\int_0^\infty f(x)e^{-x}dx \approx \frac{2-\sqrt{2}}{4}f(2+\sqrt{2}) + \frac{2+\sqrt{2}}{4}f(2-\sqrt{2})$$

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Day 2, Q4

For $s \neq 0$, solve

$$xu'' - u' + (1-x)u = \delta(x-s)$$

with boundary conditions $\lim_{x\to\pm\infty} = 0$.

Hint: e^x is a solution of the homogeneous equation.

Solution. We already have one solution $u_1(x) = e^x$ to the homogeneous equation. To find the second, try $u_2(x) = v(x)e^x$ for some yet to be determined function v. Plugging $u_2(x)$ into the ODE yields

$$\begin{aligned} xv''e^x + v'(2x-1)e^x &= 0 \\ \implies v'' + \left(2 - \frac{1}{x}\right) &= 0 \\ \implies \ln v' = \ln x - 2x + B & \\ \implies v' &= Bxe^{-2x} \\ \implies v(x) &= B\left(x - \frac{1}{2}\right)e^{-2x} \\ \implies u_2(x) &= v(x)e^x = B\left(x - \frac{1}{2}\right)e^{-x} \end{aligned}$$

In order to satisfy the boundary conditions, we need that the solution to the left of s is of the form

$$u_L(x) = C_L e^x$$

and the solution to the right of s is of the form

$$u_R(x) = C_R\left(x - \frac{1}{2}\right)e^{-x}$$

To solve the given differential equation, we need to apply the condition of continuity at $x = s$. Equating $u_L(s) = u_R(s)$ gives the relation

$$\begin{aligned} C_L e^s &= C_R\left(s - \frac{1}{2}\right)e^{-s} \\ \implies C_L &= C_R\left(s - \frac{1}{2}\right)e^{-2s} \end{aligned}$$

Now, apply the jump condition at $x = s$ of strength $\frac{1}{s}$. The jump condition is

$$\left.\frac{du_R}{dx}\right|_{x=s} - \left.\frac{du_L}{dx}\right|_{x=s} = \frac{1}{s}.$$

Using this condition gives the final relation needed to solve the equation. The coefficients are

$$\begin{aligned} C_L &= -\frac{s(s-\frac{1}{2})}{2(s-1)}e^{-s} \\ C_R &= -\frac{s}{2(s-1)}e^{s} \end{aligned}$$

The solution is

$$u(x;s) = -\frac{s(s-\frac{1}{2})}{2(s-1)}e^{x-s} - \frac{s}{2(s-1)}e^{s-x}.$$

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Day 2, Q5

Show that there is no smooth solution $u(x)$ of the following boundary value problem.

$$\begin{aligned} u_{xx} + u(1-u)(u - \frac{1}{4}) = 0,\ x \in (-\infty, \infty) \\ \lim_{x\to\infty} u(x) = 1 \\ \lim_{x\to-\infty} u(x) = 0 \\ \lim_{x\to\infty} u'(x) = 0 \\ \lim_{x\to\infty} u'(x) = 0 \end{aligned}$$

Hint: multiply the equation by u_x.

Solution. Take the hint and the equation becomes

$$\begin{aligned} u_{xx}u_x + uu_x(1-u)(u - \frac{1}{4}) = 0,\ x \in (-\infty, \infty) \\ u_x u_{xx} - u_x u^3 + \frac{5}{4}u_x u^2 - \frac{1}{4}u_x = 0. \end{aligned}$$

Use the relations $u_x u_{xx} = \frac{1}{2}\frac{d}{dx}(u_x^2)$ and $u_x u^n = \frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)$. The differential equation becomes

$$\frac{d}{dx}(u_x^2 - \frac{u^4}{4} + \frac{5}{12}u^3 - \frac{1}{8}u^2) = 0$$

Now, by the FTOC

$$\int_{-\infty}^{x}(u_x^2 - \frac{u^4}{4} + \frac{5}{12}u^3 - \frac{1}{8}u^2)dx = C.$$

From the boundary condition $\lim_{x\to\infty} u(x) = 1$, it is implied that the $C = \frac{1}{24}$. However, when we consider the boundary condition $\lim_{x\to-\infty} u(x) = 0$, it is implied that the $C = 0$. This is a contradiction.

Therefore, there is no smooth solution $u(x)$ of the given boundary value problem.

Day 2, Q6

Define a sequence of distributions by

$$T_n(\phi) = \int_{-\infty}^{\infty} \hat{\phi}(k) t_n(k) dk$$

where $t_n(k) = n^2 t(nk)$ and $t(k)$ is given by the graph below. $\hat{\phi}(k)$ is the Fourier transform of $\phi(x)$:

$$\hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx$$

Let $T = \lim_{n\to\infty} T_n$, where the limit is taken in the sense of distributions. Give an explicit formula for T.

Solution. Use a taylor expansion for $\hat{\phi}(k)$

$$\hat{\phi}(k) = \hat{\phi}(0) + k\hat{\phi}'(0) + \mathcal{O}(k^2)$$

Now, the integral for $T_n(\phi)$ becomes

$$\begin{aligned}\int_{-\infty}^{\infty} t_n(k)dk &= \int_{-\infty}^{\infty} [\hat{\phi}(0) + k\hat{\phi}'(0) + \mathcal{O}(k^2)]t_n(k)dk \\ &= \hat{\phi}'(0)\int_{-\infty}^{\infty} k t_n(k)dk + \int_{-\infty}^{\infty} \mathcal{O}(k^2)t_n(k)dk, \quad \tilde{k} = nk\end{aligned}$$

Through the variable substitution, it the second integral has a $\frac{1}{n}$ leading term. Therefore, in the limit as $n \to \infty$ the second integral goes to zero. So

$$\begin{aligned}\lim_{n\to\infty} T_n(\phi) &= \lim_{n\to\infty} 2\left[n^2 \int_0^{\frac{1}{2}} \frac{\tilde{k}}{n} 2\tilde{k} \frac{d\tilde{k}}{n} + n^2 \int_{\frac{1}{2}}^{1} \frac{\tilde{k}}{n} 2(1-\tilde{k})\frac{d\tilde{k}}{n}\right] \\ &= \frac{1}{2}\hat{\phi}'(0)\end{aligned}$$

So, in distribution, $T_n \to \hat{\delta}'(k)$.

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Day 2, Q7

For $n = 0, 1, 2, \ldots$, a function f_n on the real line is defined by

$$f_n(x) = \left(\frac{d}{dx} - x\right)^n e^{-x^2/2}$$

The f_n are eigenfunctions of a self-adjoint operator H on $L^2(\mathbb{R}, dx)$ with distinct eigenvalues λ_n. Given that the polynomials are dense in $L^2(\mathbb{R}, e^{-x^2}dx)$, show that in $L^2(\mathbb{R}, dx)$, $\left\{\frac{f_n}{||f_n||}\right\}_{n=0}^{\infty}$ is an orthonormal basis.

Here, $L^2(\mathbb{R}, dx)$ denotes the Hilbert space of real-valued functions over the real line with $\int f^2 dx < \infty$. It is equipped with the inner product $(f, g) = \int fg dx$.

Solution. Since H is self-adjoint, the following is true

$$\begin{aligned}(Hf_n, f_m) &= (\lambda_n f_n, f_m) = \lambda_n(f_n, f_m)\\ (Hf_n, f_m) &= (f_n, Hf_m) = (f_n, \lambda_m f_m) = \lambda_m(f_n, f_m)\\ \implies (\lambda_n - \lambda_m)(f_n, f_m) = 0\end{aligned}$$

But, since H has distinct eigenvalues, for $m \neq n$ $(f_n, f_m) = 0$. So the set $\{f_n\}$ is orthogonal. By definition then $\left\{\frac{f_n}{||f_n||}\right\}_{n=0}^{\infty}$ is an orthonormal set. Now, we need to show that this set is a basis.

To do this, first recall that polynomials are dense in $L^2(\mathbb{R}, e^{-x^2}dx)$, given $\phi \in L^2(\mathbb{R}, e^{-x^2}dx)$ there is a polynomial P such that $||\phi - P||_{L^2(\mathbb{R}, e^{-x^2}dx)} < \epsilon$. Let $N = \deg(P)$.

$$f_n(x) = p_n(x)e^{-x^2/2}, \quad \text{where } p_n \text{ is an } n\text{th degree polynomial}$$

so $p_1, \ldots, p_n$ form a linearly independent set. With this set, we can expand the polynomial P in terms of elements of the set.

$$P = \sum_{k=1}^{N} a_k p_k$$

So, given a $\phi \in L^2(\mathbb{R}, e^{-x^2}dx)$, and $\epsilon > 0$, there is an $N \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ s.t.

$$||\phi - \sum_{k=1}^{N} a_k p_k||_{L^2(\mathbb{R}, e^{-x^2}dx)} < \epsilon.$$

then

$$\begin{aligned}\int_{-\infty}^{\infty}|\phi(x)-\sum_{k=1}^{N}a_kp_k|^2e^{-x^2} &< \epsilon^2\\ \int_{-\infty}^{\infty}|\phi(x)e^{-x^2/2}-\sum_{k=1}^{N}a_kp_ke^{-x^2/2}|^2 &< \epsilon^2\end{aligned}$$

$$||\phi(x)e^{-x^2/2}-\sum_{k=1}^{N}a_kp_ke^{-x^2/2}||_{L^2(\mathbb{R})}<\epsilon$$

But, since $\phi\in L^2(\mathbb{R},e^{-x^2}dx)$ then $\phi e^{-x^2}\in L^2(\mathbb{R},dx)$. Therefore, given any $\psi\in L^2(\mathbb{R},dx)$, and $\epsilon>0$, there exists $N\in\mathbb{N}$ and $a_1,a_2,\ldots,a_N\in\mathbb{R}$ s.t.

$$\left|\left|\psi-\sum_{k=1}^{N}a_kp_ke^{-x^2/2}\right|\right|_{L^2(\mathbb{R})}<\epsilon$$

Therefore, given any function in $L^2(\mathbb{R},dx)$, it can be represented by a linear combination of elements from the orthonormal set $\{f_n\}$. Hence, $\left\{\frac{f_n}{||f_n||}\right\}_{n=0}^{\infty}$ is an orthonormal basis.

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Day 2, Q8

Show that for every real $N\times N$ matrix A there exists an invertible real $N\times N$ matrix U and a real block-diagonal matrix B such that

$$U^{-1}AU=B=\begin{pmatrix}B_1&&&&\\&B_2&&&\\&&\ddots&&\\&&&B_{m-1}&\\&&&&B_m\end{pmatrix}$$

where each block B_k has either the form

$$B_k=\begin{pmatrix}\lambda_k&1&&&\\&\lambda_k&1&&\\&&\ddots&\ddots&\\&&&\lambda_k&1\\&&&&\lambda_k\end{pmatrix}$$

for some real λ_k, or the form

$$B_k = \left(\begin{array}{cccccccccc}
\alpha_k & \beta_k & 1 & 0 & & & & & & \\
-\beta_k & \lambda_k & 0 & 1 & & & & & & \\
 & & \alpha_k & \beta_k & 1 & 0 & & & & \\
 & & -\beta_k & \alpha_k & 0 & 1 & & & & \\
 & & & & \ddots & & \ddots & & & \\
 & & & & & \ddots & & \ddots & & \\
 & & & & & & \alpha_k & \beta_k & 1 & 0 \\
 & & & & & & -\beta_k & \alpha_k & 0 & 1 \\
 & & & & & & & & \alpha_k & \beta_k \\
 & & & & & & & & -\beta_k & \alpha_k
\end{array}\right)$$

for some real α_k and β_k with $\beta_k \neq 0$. All matrix elements which are not shown are zero. Hint: consider the Jordan normal form.

Solution. For an eigenvalue whose geometric multiplicity is equal to the algebraic multiplicity, the Jordan block does not have ones along the super-diagonal...strictly diagonal, otherwise the ones will be present along the super-diagonal.

Now suppose that we don't have access to the complex numbers.

If A is a real matrix, then any complex eigenvalues and eigenvectors come in conjugate pairs.

If A is real and $\lambda = \alpha + i\beta$ is an eigenvalue with eigenvector u, then $\bar{\lambda} = \alpha - i\beta$ is an eigenvalue with eigenvector $\bar{u}$.

Suppose A is real and $Au = (\alpha + i\beta)u$.

$$\begin{array}{rcl}
A(u_R + iu_I) & = & (\alpha + i\beta)(u_R + iu_I) \\
Au_R + iAu_I & = & (\alpha u_R - \beta u_I) + i(\beta u_R + \alpha u_I) \\
 & \downarrow & \\
Au_R & = & \alpha u_R - \beta u_I \quad \text{and} \\
Au_I & = & \beta u_R + \alpha u_I \\
A(u_R \;\; u_I) & = & (\alpha u_R - \beta u_I \;\;\; \beta u_R + \alpha u_I) \\
 & = & (u_R \;\; u_I) \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}
\end{array}$$